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A Class of Bounded Hereditary Noetherian Domains

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Let k be a commutative field, F and G division rings containing k in the center and finite dimensional over k. Let ${}_F M_G$ be a bimodule, with k operating centrally, and such that $\dim {}_F M = \dim M_G = 2$. Assume, the element $m \in M$ generates M as a bimodule. We are going to define a k-algebra $R(m)$ as follows: Let $F *_k G$ be the free product of F and G over k, let $I(m)$ be the ideal of $F *_k G$ generated by all elements of the form $\sum_{i=1}^n f_i * g_i$, with $\sum_{i=1}^n f_i m g_i = 0$ in M. Then, by definition, $R(m) = F *_k G/I(m)$.

THEOREM. *The k-algebra $R(m)$ is infinite dimensional, and it is a bounded hereditary noetherian domain.*

For the convenience of the reader, let us recall the definitions. A ring R without proper zero divisors is called a domain. The ring R is said to be noetherian provided it satisfies the ascending chain condition both for left ideals and for right ideals. It is called hereditary, in case submodules of projective modules are projective, and finally, a noetherian domain R is said to be bounded provided any non-zero left or right ideal contains a non-zero two-sided ideal.

We are indebted to the referee for his or her valuable remarks; in particular, a result of the forthcoming book "Simple Artinian Rings" by A. H. Schofield has been brought to our attention asserting that the algebra $R(m)$ is a fir (that is, that every one-sided ideal is a free module with

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invariant rank). In combination with our theorem, this implies that $R(m)$ is, in fact, a principle ideal domain.

In the course of the proof of the theorem, we will encounter further properties of the algebra $R(m)$. Actually, we will show that any non-zero one-sided ideal of $R(m)$ is of finite codimension over k; of course this immediately will imply that $R(m)$ is noetherian and bounded. Consequently, all simple $R(m)$-modules are finite dimensional over k. Also, given two simple $R(m)$-modules S, S', we will see that $\mathrm{Ext}^1(S, S') \neq 0$ iff S and S' are isomorphic. Thus, any proper factor ring of $R(m)$ is uniserial (in the sense of Nakayama: a direct product of a finite number of full matrix rings over local serial rings—in general, it is known that proper factor rings of hereditary noetherian prime rings are serial, but not necessarily uniserial, a theorem of Eisenbud, Griffith and Robson [6]).

Let us mention two types of examples. First, let F, G be commutative fields, with a common subfield k of index 2 both in F and in G. Then the free product $F *_k G$ of F and G over k is of the form $R(m)$ (for $M = {}_F(F \otimes G)_G$, and $m = 1 \otimes 1 \in M$). This seems to be of interest both in case $F = G$ as well as in case that F, G are non-isomorphic. Second, let F be an arbitrary division ring, finite dimensional over k. Then the canonical bimodule ${}_F(F \oplus F)_F$ has a generator if and only if F is not commutative.

For the proof of the theorem, we will construct an embedding functor Φ from the category $\mathscr{M}_{R(m)}$ of right $R(m)$-modules into the category $\mathscr{M}({}_F M_G)$ of representations of the bimodule ${}_F M_G$. We will use the structure theory of $\mathscr{M}({}_F M_G)$ as developped in [3, 7–9] in order to derive the various properties of the ring $R(m)$. Of particular importance in $\mathscr{M}({}_F M_G)$ is some infinite-dimensional representation Q of ${}_F M_G$ with $E = \mathrm{End}(Q)$ a division ring and ${}_E Q$ finitely generated. It turns out that E is the quotient division ring of $R(m)$, and $Q = \Phi(E_{R(m)})$.

In the present paper, properties of the category $\mathscr{M}({}_F M_G)$ are exploited in order to obtain information about the ring $R(m)$. We should point out that our own interest in the ring $R(m)$ lies in the fact that conversely $R(m)$ should give information about the category $\mathscr{M}({}_F M_G)$. Namely, the maximal spectrum of $R(m)$ is a convenient index set for the set of isomorphism classes of simple regular representations of ${}_F M_G$ which are different from the fixed representations S_m (see Section 2 below). Given an arbitrary tame finite-dimensional hereditary algebra, all indecomposable representations but the homogeneous ones have been determined in [3, 5] and all of them have been shown to be characterized by combinatorial invariants. On the other hand, the description of the homogeneous representations cannot be purely combinatorial. The problem of describing the category of all homogeneous representations of a tame finite-dimensional hereditary k-algebra has been reduced in [3, 5] to the special case of the tensor algebra $T({}_F M_G)$ of a bimodule ${}_F M_G$, with F, G division rings containing k in the center and finite dimensional over k, and k operating centrally on ${}_FM_G$, such that moreover $(\dim {}_FM)(\dim M_G) = 4$. In case the bimodule ${}_FM_G$ is not simple, the homogeneous representations of ${}_FM_G$ can be described using modules over a twisted polynomial ring, see [7]. Our present paper now deals with the case where ${}_FM_G$ is a simple bimodule and $\dim {}_FM = \dim M_G = 2$.

1. Embedding of $\mathcal{M}_{R(m)}$ into the Category of Representations of ${}_FM_G$

Recall the definition of a representation (X_F, Y_G, φ) of the bimodule ${}_FM_G$; it is given by an F-vectorspace X_F, a G-vectorspace Y_G and a linear transformation $\varphi: X_F \otimes_F M_G \to Y_G$. A map from (X_F, Y_G, φ) to (X'_F, Y'_G, φ') is of the form (f, g), with $f: X_F \to X'_F$, $g: Y_G \to Y'_G$ linear transformations satisfying $g\varphi = \varphi'(f \otimes 1_M)$. The category of representations of ${}_FM_G$ is denoted by $\mathcal{M}({}_FM_G)$.

Now consider the k-algebra $R(m)$. The residue class of $f * g \in F *_k G$ modulo $I(m)$ will be denoted by $\overline{f * g}$. Note that these elements generate $R(m)$.

Given an $R(m)$-module $X_{R(m)}$, we may consider X both as an F-vectorspace as well as a G-vectorspace, using the ring homomorphisms $F \to R(m)$, $f \mapsto \overline{f * 1}$ and $G \to R(m)$, $g \mapsto \overline{1 * g}$. It will be clear below that these maps are in fact inclusion maps, as soon as we know that $R(m)$ is a non-zero ring. Also, the $R(m)$-module structure on X defines a linear transformation

$$\varphi_X : X_F \otimes_F M_G \to Y_G, \qquad \varphi_X\left(x \otimes \sum_{i=1}^{n} f_i m g_i\right) = \sum_{i=1}^{n} x \cdot \overline{f_i * g_i},$$

where $x \in X$, and all $f_i \in F$, $g_i \in G$. Here, we use that m generates the bimodule ${}_FM_G$, so that any element of M is of the form $\sum_{i=1}^{n} f_i m g_i$, and φ_X is well defined due to the definition of $I(m)$. In this way, we obtain from $X_{R(m)}$ a representation (X_F, X_G, φ_X) of ${}_FM_G$. Also, given an $R(m)$-homomorphism $f: X_{R(m)} \to X'_{R(m)}$, it is obvious that (f, f) is a map from (X_F, X_G, φ_X) to $(X'_F, X'_G, \varphi_{X'})$ in $\mathcal{M}({}_FM_G)$; thus we obtain a functor $\Phi: \mathcal{M}_{R(m)} \to \mathcal{M}({}_FM_G)$, which obviously is exact, and commutes with arbitrary direct sums.

1.1. Proposition. *The functor Φ gives an equivalence between the category $\mathcal{M}_{R(m)}$ and the full subcategory $\mathcal{R}(m)$ of all representations (X_F, Y_G, φ) of ${}_FM_G$ with $\varphi(- \otimes m): X \to Y$ being bijective.*

Note that the map $\varphi(- \otimes m)$ is k-linear, but may not respect any other module structure.

Proof of Proposition 1.1. It is obvius that the image of Φ is contained in $\mathscr{R}(m)$, since for any $R(m)$-module $X_{R(m)}$, and any $x \in X$, we have $\varphi_X(x \otimes m) = x \cdot \overline{1 * 1} = x$, thus the map $\varphi_X(- \otimes m): X \to X$ is the identity. Conversely, given any representation (X_F, Y_G, φ) of ${}_FM_G$ which belongs to $\mathscr{R}(m)$, we may construct an $F * G$-module structure on X as follows: given $x \in X, f \in F$, $g \in G$, let $x \cdot (f * g)$ be defined by the equation

$$\varphi(x \cdot (f * g) \otimes m) = \varphi(x \otimes fmg).$$

Note that $x \cdot (f * g)$ is uniquely determined due to our assumption that (X_F, Y_G, φ) belongs to $\mathscr{R}(m)$. It is obvious that X as an $F * G$-module is annihilated by $I(m)$; thus, in this way, X becomes an $R(m)$-module, and clearly $\Phi(X_{R(m)})$ is isomorphic to (X_F, Y_G, φ).

1.2. COROLLARY. *The ring $R(m)$ is hereditary.*

Proof. The category $\mathscr{M}({}_FM_G)$ is hereditary, and $\mathscr{R}(m)$ is closed under extension, thus it is hereditary, too. (An abelian category is said to be hereditary provided the functor Ext^1 is right exact in the second variable; of course, the category of modules over a ring is hereditary if and only if the ring is hereditary.)

1.3. We may give another interpretation to the functor Φ. Namely, $\mathscr{M}({}_FM_G)$ may be considered as the category of $T({}_FM_G)$-modules, where $T({}_FM_G)$ is the matrix ring $\left(\begin{smallmatrix} F & M \\ 0 & G \end{smallmatrix}\right)$. Now, Φ is the composition of the usual Morita equivalence functor from $\mathscr{M}_{R(m)}$ to $\mathscr{M}_{M_2(R(m))}$, where $M_2(R)$ denotes the ring of 2×2-matrices over R, with a full embedding functor $\mathscr{M}_{M_2(R(m))} \to \mathscr{M}_{T({}_FM_G)}$. This full embedding functor is induced by a ring epimorphism

$$\varepsilon: T({}_FM_G) \to M_2(R(m)).$$

We will see later that this ring epimorphism is in fact a monomorphism.

2. FINITE-DIMENSIONAL $R(m)$-MODULES

Note that the ring $R(m)$ is a k-algebra; we denote by $m_{R(m)}$ the full subcategory of all $R(m)$-modules which are finite-dimensional k-vectorspaces. Under Φ, the category $m_{R(m)}$ is mapped onto the full subcategory $r(m)$ of all finite-dimensional representations in $\mathscr{R}(m)$.

We recall that a finite-dimensional representation of ${}_FM_G$ is said to be *regular* provided it is the direct sum of indecomposable representations of the form (X_F, Y_G, φ) with $\dim X_F = \dim Y_G$, and we denote the full subcategory of all regular finite-dimensional representations of ${}_FM_G$ by r.

Now $\mathscr{I}$ is an abelian category [3], so we may speak of simple objects in $\mathscr{I}$, composition series in $\mathscr{I}$, and so on. We recall from [7] that any indecomposable object in $\mathscr{I}$ has a unique composition series in $\mathscr{I}$, and all its composition factors in $\mathscr{I}$ are isomorphic. Conversely, given any simple object in $\mathscr{I}$, there are indecomposable objects in $\mathscr{I}$ of arbitrarily large length having it as composition factor. (Thus, $\mathscr{I}$ is a direct sum of categories each of which is serial with a unique simple object.)

Given $0 \neq a \in M$, one may construct a simple object S_a in $\mathscr{I}$ as follows: let

$$S_a = (F_F, G_G, \pi_a : F_F \otimes {}_F M_G \approx M_G \to M_G/aG \approx G_G)$$

using the canonical projection of M_G onto M_G/aG. We denote by $\mathscr{U}(a)$ the full subcategory of $\mathscr{I}$ of all objects with composition factors in $\mathscr{I}$ being of the form S_a. We will need the following lemma.

2.1. LEMMA. *Let (X_F, Y_G, φ) be a representation of ${}_F M_G$. Then $\varphi(- \otimes m): X \to Y$ is injective if and only if* $\operatorname{Hom}(S_m, (X_F, Y_G, \varphi)) = 0$.

Proof. If $(f, g): S_m \to (X_F, Y_G, \varphi)$ is a non-zero map, then $f \neq 0$, since π_m is surjective and $g\pi_m = \varphi(f \otimes 1_M)$. However, $\varphi(f(1) \otimes m) = g\pi_m(1 \otimes m) = 0$, thus $f(1)$ is a non-zero element belonging to the kernel of $\varphi(- \otimes m): X \to Y$.

Conversely, assume there exists $0 \neq x \in X$ with $\varphi(x \otimes m) = 0$. Define a homomorphism $(f, g): S_m \to (X_F, Y_G, \varphi)$ by $f(1) = x$, and $g(1) = m'$, where m' is an element of M with $\pi_m(1 \otimes m') = 1 \in G$. Thus $\operatorname{Hom}(S_m, (X_F, Y_G, \varphi)) \neq 0$.

2.2. COROLLARY. $\mathscr{I} = \mathscr{I}(m) \amalg \mathscr{U}(m)$.

Proof. An indecomposable object (X_F, Y_G, φ) in $\mathscr{U}(m)$ has a subobject of the form S_m, thus $\operatorname{Hom}(S_m, (X_F, Y_G, \varphi)) \neq 0$, and by the lemma, (X_F, Y_G, φ) cannot belong to $\mathscr{I}(m)$.

On the other hand, if (X_F, Y_G, φ) is an indecomposable object in $\mathscr{I}$ and not belonging to $\mathscr{U}(m)$, then $\operatorname{Hom}\ (S_m, (X_F, Y_G, \varphi)) = 0$, thus $\varphi(- \otimes m)$ is injective. Since $\dim_k X = \dim_k Y$ due to the fact that (X_F, Y_G, φ) belongs to $\mathscr{I}$, the k-linear map $\varphi(- \otimes m)$ is bijective.

2.3. COROLLARY. *Any indecomposable finite-dimensional $R(m)$-module has a unique composition series with all composition factors being isomorphic, and given any simple-dimensional $R(m)$-module, there exist indecomposable $R(m)$-modules of arbitrarily large length having this module as composition factor.*

Note that, up to now, we do not yet know whether there are any non-zero finite-dimensional $R(m)$-modules. In fact, we do not even know whether $R(m)$ may not be the zero ring.

2.4. LEMMA. *There exists a non-zero simple $R(m)$-module X with* $\dim_k X = \dim_k F$.

Proof. Following [2], we show that $FmG = \{fmg \mid f \in F,\ g \in G\}$ is a proper subset of M. Consider the group $F^\times \times G^\times$ which operates on M via $(f, g)\, x = fxg^{-1}$, and not that the diagonal $\Delta = \{(t, t) \mid t \in k^\times\} \subseteq F^\times \times G^\times$ operates trivially. Thus $F^\times \times G^\times/\Delta$ operates on M. Obviously, $FmG \backslash \{0\}$ is an $F^\times \times G^\times/\Delta$-orbit. In case k is finite we count elements: denote the number of elements of a set S by $|S|$, and let $n = |F|$; then also $|G| = n$, and $|M| = n^2$, due to the fact that $\dim_F M = \dim M_G = 2$. Then

$$\begin{aligned} |FmG| &= |FmG\backslash\{0\}| + 1 \leqslant |F^\times \times G^\times/\Delta| + 1 \\ &= \frac{(n-1)^2}{|k^\times|} + 1 \leqslant n^2 - 2n + 2 < n^2. \end{aligned}$$

Similarly, in case k is infinite, we count dimensions. This is possible, since we deal with an action of the algebraic group $F^\times \times G^\times/\Delta$ on the affine algebraic variety M. Let $d = [F\colon k]$, then

$$\operatorname{Dim}(F^\times \times G^\times/\Delta) = 2d - 1 < 2d = \operatorname{Dim} M,$$

thus there are even infinitely many $F^\times \times G^\times/\Delta$-orbits on M. Thus, also in this case FmG is a proper subset of M.

We claim that for any $x \in M\backslash FmG$, the representation S_x belongs to $R(m)$. Namely, if $x \in M\backslash FmG$, then $Fm \cap xG = 0$, and consequently $Fm \oplus xG = M$, since Fm and xG are k-linear subspaces. However, this obviously implies that for the projection map $\pi_x\colon F_F \otimes {}_F M_G \approx M_G \to M_G/xG \approx G_G$, the restriction to $F \otimes m$ is bijective.

2.5. COROLLARY. $R(m)$ is an infinite-dimensional k-algebra.

Proof. The lemma aserts the existence of at least one simple finite-dimensional $R(m)$-module X. By the previous corollary, there exist indecomposable modules with X as composition factor, and having a unique composition series of arbitrarily large length. Since any such module is monogenic, thus a factor module of $R(m)_{R(m)}$, it follows that $R(m)$ cannot be finite dimensional.

2.6. COROLLARY. *The elements $\overline{f * 1}$, $f \in F$, form a division subring of $R(m)$, canonically isomorphic to F; the elements $\overline{1 * g}$, $g \in G$, form a division subring of $R(m)$, canonically isomorphic to G. In this way, $R(m)$ may be considered as an $F-G$-bimodule, and the $F-G$-subbimodule of ${}_F R(m)_G$ generated by $\overline{1 \otimes 1}$ is isomorphic to ${}_F M_G$.*

Proof. The first two assertions follow directly from the fact that $\overline{1 * 1}$ is non-zero. Now consider the bimodule map $\mu: {}_F M_G \to {}_F R(m)_F$ given by $m \mapsto \overline{1 \otimes 1}$ (it is well defined due to the definition of $I(m)$). The image $\bar{M}$ of μ is non-zero, thus its left F-dimension is either one or two. However, if this left F-dimension would be one, then it would coincide with any one of the division subrings $\bar{F} = \{\overline{f * 1} \mid f \in F\}$ and $\bar{G} = \{\overline{1 * g} \mid g \in G\}$. Since these two division subrings generate $R(m)$ as a ring, it follows from $\bar{F} = \bar{G}$ that $R(m) = \bar{F}$ is finite dimensional over k, a contradiction. This shows that μ is a monomorphism.

2.7. It follows that the ring epimorphism ε given in 1.3 is a monomorphism. In fact, we may identify $T({}_F M_G)$ with the subring of $M_2(R(m))$ given by

$$\begin{pmatrix} \bar{F} & \bar{M} \\ 0 & \bar{G} \end{pmatrix},$$

where we use the notation of 2.6.

3. $R(m)$ Is a Subring of a Division Ring

We are going to use the structure theory for infinite-dimensional representations of ${}_F M_G$ as developped in [8] in order to derive further properties of $R(m)$. Recall that an indecomposable finite-dimensional representation (X_F, Y_G, φ) of ${}_F M_G$ is said to be preprojective, regular, or preinjective, provided $\dim X_F - \dim Y_G$ is <0, $=0$, or >0, respectively. The (not necessarily finite-dimensional) representation A of ${}_F M_G$ is said to be *torsion* provided it is generated by finite-dimensional representations which are regular or preinjective, and *torsionfree* provided $\operatorname{Hom}(B, A) = 0$ for any finite-dimensional representation B which is regular or preinjective. Also, A is said to be regular provided it does not have an indecomposable finite-dimensional direct summand which is preprojective or preinjective. Note that $\mathscr{R}(m)$ contains only regular modules. (Namely, $\mathscr{R}(m)$ is closed under direct summands, now use 2.2.)

3.1. Lemma. *$\Phi(R(m)_{R(m)})$ is torsionfree regular.*

Proof. First, assume $\Phi(R(m)_{R(m)})$ has an indecomposable direct summand B of finite length. Then B has to be regular, say with regular socle C. There exists an exact sequence

$$0 \to C \to B' \to B \to 0 \qquad (*)$$

with B' indecomposable. With B also C, and therefore also B' belongs to $\mathscr{R}(m)$. This shows that B cannot be a projective object of the abelian category $\mathscr{R}(m)$, in contrast to the fact that B is a direct summand of the projective object $\Phi(R(m)_{R(m)})$. Similarly, we see that $\Phi(R(m)_{R(m)})$ cannot have a direct summand B which is a Prüfer module. (Prüfer modules are infinite-dimensional indecomposable representations which are unions of countable chains of finite-dimensional indecomposable regular representations.) Namely, in this case there exists an exact sequence $(*)$ with B' isomorphic to B, thus again B cannot be projective in $\mathscr{R}(m)$. Our assertion now follows from Proposition 4.8 of [8].

Recall that there exists a unique infinite-dimensional representation Q of ${}_FM_G$ with $E = \operatorname{End}(Q)$ being a division ring and such that Q is finitely generated over E (Theorems 5.3 and 5.7 of [8], and the main result of [9]). In fact, $\dim_E Q = 2$, according to 5.7 of [8]. Also note that Q is torsionfree regular and has no non-trivial fully invariant submodule.

3.2. LEMMA. *Q belongs to $\mathscr{R}(m)$.*

Proof. Let $Q = (Q'_F, Q''_G, \eta)$ with $\eta: Q'_F \otimes {}_FM_G \to Q''_G$. Now Q is torsionfree, thus $\operatorname{Hom}(S_m, Q) = 0$, thus $\eta(- \otimes m)$ is injective.

Assume $\eta(- \otimes m)$ is not surjective; say, assume there is $q \in Q''$, not in the image of $\eta(- \otimes m)$. There exists a finite-dimensional subrepresentation $A = (A'_F, A''_G, \eta)$ of Q with $q \in A''$. (This is a direct consequence of the construction of Q in 5.2 and 5.3 of [8].) We denote by $P(0) = (0, G_G, o)$ and $P(1) = (F_F, M_G, 1_M)$ the two indecomposable projective representations. The homomorphisms $P(0) \to (X_F, Y_G, \varphi)$ are of the form $\alpha_y = (\alpha'_y, \alpha''_y)$ with $y \in Y$, where $\alpha'_y = o: 0 \to X_F$, and $\alpha''_y: G_G \to Y_G$, $\alpha''_y(g) = yg$. Consider the kokernel of $\binom{\alpha_m}{\alpha_q}: P(0) \to P(1) \oplus B$, say, $(\beta\gamma): P(1) \oplus A \to B$. Note that γ cannot be split mono. (Otherwise, we can assume $\gamma = \binom{0}{1}: A \to S \otimes A$ and write $\beta = \binom{\beta_1}{\beta_2}$ with $\beta_1: P(1) \to S$, $\beta_2: A \to S$. It follows that $\beta_2\alpha_m + \alpha_q = 0$. Write $\beta_2 = (\beta'_2, \beta''_2)$, thus $q = \alpha''_y(1) = -\beta''_2\alpha''_m(1) = \beta''_2(-m)$, and therefore $\eta(-\beta'_2(1) \otimes m) = \beta''_2(-m) = q$ contradicts our assumption on q.) Now the defect of B is -1, and all indecomposable representations C of ${}_FM_G$ with defect -1 and with a non-zero, non-invertible map $A \to C$ have dimension vector $\geqslant \dim B$, thus B is indecomposable. Also, γ is a monomorphism, and we may suppose that γ is, in fact, an inclusion. Thus, we consider A as a subrepresentation of B. The inclusion of A into Q extends to a map δ from B into Q, since B/A is regular, and δ is a monomorphism (since otherwise the image of δ would be a subrepresentation of Q of defect $\geqslant 0$). Thus, we can assume $A \subseteq B \subseteq Q$, and by the construction of $B = (B', B'', \eta)$ there exists $x \in B'$ with $\eta(x \otimes m) = q$, in contrast to our assumption on q. Thus $\eta(- \otimes m)$ is surjective on Q, and therefore Q belongs to $\mathscr{R}(m)$.

3.3. PROPOSITION. *$R(m)$ is a subring of E and* $\operatorname{End}(E_{R(m)}) = E$.

Proof. Since Q is in $\mathscr{R}(m)$, there exists a right module $I_{R(m)}$ with

$\Phi(I_{R(m)}) = Q$. We have $E = \mathrm{End}(Q) = \mathrm{End}(I_{R(m)})$, and $\dim_E Q = 2$ implies that $\dim_E I = 1$. Now any embedding of Q into a regular representation of ${}_FM_G$ splits, thus Q is an injective object of $\mathscr{R}(m)$ and I is an injective $R(m)$-module. We have seen above that $\Phi(R(m)_{R(m)})$ is torsionfree, thus it embeds into a direct sum of copies of Q (5.5 of [8]), and therefore $R(m)_{R(m)}$ can be embedded into a direct sum of copies of I. This shows that $I_{R(m)}$ is faithful. As a consequence, the double centralizer map $R(m) \to \mathrm{End}({}_EI)$ is a monomorphism of rings. Since $\dim {}_EI = 1$, we can identify $\mathrm{End}({}_EI)$ with E, thus $R(m)$ is embedded into E. Using this identification, the module $E_{R(m)}$ is isomorphic to $I_{R(m)}$.

Observe that if m' is another generator of the bimodule M and $R(m')$ is the corresponding ring, then the skew fields of fractions of $R(m)$ and $R(m')$ are isomorphic, since they are both the endomorphism ring of the unique representation Q of ${}_FM_G$.

4. Proof of the Theorem

4.1. Proposition. *Any non-zero right ideal of $R(m)$ has finite codimension in $R(m)$.*

Proof. We have seen above that $R(m)$ is a subring of a division ring E, such that $\mathrm{End}(E_{R(m)}) = E$ operating by left multiplication on E, and $\Phi(E_{R(m)}) \approx Q$. The exact sequence

$$0 \to R(m)_{R(m)} \to E_{R(m)} \to (E/R(m))_{R(m)} \to 0$$

gives under Φ an exact sequence

$$0 \to \Phi(R(m)_{R(m)}) \to Q \to \Phi(E/R(m)) \to 0$$

in $R(m)$. According to Corollary 6.1 of [8], $\Phi(E/R(m))$ is a torsion regular module, thus we see that $\Phi(R(m))$ is a torsionfree rank 1 module.

Let $U_{R(m)}$ be a non-zero right ideal of $R(m)$. The exact sequence

$$0 \to U_{R(m)} \to R(m)_{R(m)} \to R(m)/U \to 0$$

gives under Φ an exact sequence

$$0 \to \Phi(U_{R(m)}) \to \Phi(R(m)_{R(m)}) \to \Phi(R(m)/U) \to 0,$$

and again by Corollary 6.1 of [8], $\Phi(R(m)/U)$ is a torsion regular module. Of course, all simple regular composition factors of $\Phi(R(m)/U)$ belong to $\imath(m)$, thus $\Phi(R(m)/U$ can be embedded into a direct sum of Prüfer modules P_i (see [8, 4.5]) which belong to $\mathscr{R}(m)$, say,

$$\Phi(R(m)/U) \subseteq \bigoplus_{i \in I} P_i. \qquad (*)$$

Since any P_i belongs to $\mathscr{R}(m)$, it is of the form $\Phi(X_i)$, for some $R(m)$-module X_i, and X_i is the union of a countable chain of finite-dimensional submodules. The inclusion $(*)$ corresponds to an inclusion

$$R(m)/U \subseteq \bigoplus_{i \in I} X_i.$$

Since $R(m)/U$ is monogenic, we may suppose that the index set I is finite, and since X_i is the union of a chain of finite-dimensional submodules, the projection of $R(m)/U$ into any X_i is finite dimensional. Thus $R(m)/U$ itself is finite dimensional.

4.2. Corollary. *$R(m)$ is a bounded noetherian domain.*

Proof. We have seen in Section 3 that $R(m)$ is a domain. If $U_1 \subseteq U_2 \subseteq \cdots$ is an ascending chain of right ideals, we may suppose that $U_1 \neq 0$, thus U_1 is of finite codimension and the sequence has to stop after a finite number of steps. Also, given a non-zero right ideal $U_{R(m)}$ of $R(m)$, consider the annihilator J of $V_{R(m)} = (R(m)/U)_{R(m)}$. It is a two-sided ideal, and a subring of the double centralizer $\mathrm{End}({}_{\mathrm{End}(V_{R(m)})} V)$. However, since V is finite dimensional over k, the same is true for $\mathrm{End}({}_{\mathrm{End}(V_{R(m)})} V)$, thus $R(m)/J$ is finite dimensional, and therefore $J \neq 0$.

Thus, we have seen that $R(m)$ is right bounded right noetherian. Since the construction of $R(m)$ is left-right symmetric, we conclude that $R(m)$ also is left bounded left noetherian.

5. Examples

In order to deal with specific examples, we look at the ideal $I(m)$ of $F * G$. Considering ${}_FM_G$ as a right $F^{\mathrm{op}} \otimes_k G$-module with a generator m, $M_{F^{\mathrm{op}} \otimes_k G}$ is isomorphic to $F^{\mathrm{op}} \otimes_k G/L$ with a right ideal L. Given $r \in F^{\mathrm{op}} \otimes_k G$, say, $r = \sum_j f_j \otimes g_j$ with $f_j \in F$, $g_j \in G$, write $\tilde{r} = \sum_j f_j * g_j \in F * G$ (which is well defined). Now, it is clear that if $\{r_1, r_2, ..., r_s\}$ is a generating set of L (as a right $F^{\mathrm{op}} \otimes_k G$-module), then $I(m)$ is generated, as a two-sided ideal, by the set $\{\tilde{r}_1, \tilde{r}_2, ..., \tilde{r}_s\}$. In particular, if $F^{\mathrm{op}} \otimes_k G$ is semisimple (which is the case, for instance, when F and G are separable over k), then any one-sided ideal is generated by a single element; thus, $I(m)$ is generated by a single relation.

5.1. Let F, G be commutative fields with a common subfield k of index 2 both in F and in G. Let ${}_FM_G = {}_F(F \otimes_k G)_G$, and $m \in M$ any generator of ${}_FM_G$, for example, $m = 1 \otimes 1$. Then $R(m) = F *_k G$.

Note that for $F = G$, and $\mathrm{char}\, k \neq 2$, the field F has a unique automorphism σ fixing k, and ${}_F(F \otimes_k F)_F$ is the direct sum of a submodule of the form ${}_F F_F$ and one of the form ${}_F F_{\sigma F}$ (the left action being given by multiplication, the right by twisted multiplication, the twist being σ). Since these two subbimodules are non-isomorphic , all elements outside these two subbimodules are generators of ${}_F(F_k \otimes_k F)_F$. (The case of $k = \mathbb{R}$, $F = \mathbb{C}$ has been considered in detail in [4].)

5.2. Let F be a division ring, finite dimensional over k. Then the canonical bimodule ${}_F(F \oplus F)_F$ has a generator if and only if F is not commutative. Of course, if F is commutative, any non-zero element of $F \oplus F$ generates a subbimodule of the form ${}_F F_F$. If F is not commutative, take an element f outside the center of F, then it is easy to see that $(1, f) \in F \oplus F$ is a generator of the bimodule.

5.3. Other examples may be constructed as follows: Let H be a commutative field with two subfields F, G of index 2 such that also $F \cap G$ has finite index in H. Let $k = F \cap G$, and ${}_F M_G = {}_F H_G$. For example, let H be the splitting field of $X^3 - 2$ over $\mathbb{Q}$, and ζ a primitive third root of unity. We can take $F = \mathbb{Q}[\sqrt[3]{2}]$, and $G = \mathbb{Q}[\zeta \cdot \sqrt[3]{2}]$, and obtain for $m = 1 \in H$ the $\mathbb{Q}$-algebra

$$R(m) = \mathbb{Q}[\sqrt[3]{2}] * \mathbb{Q}[\zeta \sqrt[3]{2}] / \langle \sqrt[3]{4} * 1 + \sqrt[3]{2} * \zeta \sqrt[3]{2} + 1 * \zeta^2 \sqrt[3]{4} \rangle.$$

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